SOME RHEOLOGICAL PROPERTIES OF REINER – RIVLIN FLUIDS

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It is shown that the Reiner–Rivlin rheological relation is unsuitable for describing the effect of normal stresses.

Within the framework of the principle of the maximum rate of entropy production, the nonequilibrium part of the stresses in a Reiner–Rivlin fluid ([1], pp. 60-62, 780-782)

$$\sigma_k^i = -\lambda\delta_k^i + 2\mu e_k^i + 4\eta e_\alpha^i e_k^\alpha \tag{1}$$

can also be expressed as follows ([2], pp. 97-105):

$$\sigma_k^i = \frac{\rho\Phi}{2I_2G}\left[2I_1\frac{\partial G}{\partial\alpha}\delta_k^i + \left(G - \alpha\frac{\partial G}{\partial\alpha} - 3\beta\frac{\partial G}{\partial\beta}\right)e_k^i + 2\frac{\partial G}{\partial\beta}\frac{I_3}{I_2^2}e_\alpha^i e_k^\alpha\right], \tag{2}$$

where $\alpha = I_1^2/I_2$; $\beta = I_3^2/I_2^3$; $I_1 = e_\alpha^\alpha$, $I_2 = e_\beta^\alpha e_\alpha^\beta$; $I_3 = e_\beta^\alpha e_\gamma^\beta e_\alpha^\gamma$; and $G(\alpha, \beta)$ is an arbitrary function.

We will investigate the case of an incompressible fluid, but we will first write the spur and the deviator of the tensor (2):

$$\sigma_\alpha^\alpha = \frac{\rho\Phi}{2I_2G}\left[6I_1\frac{\partial G}{\partial\alpha} + \left(G - \alpha\frac{\partial G}{\partial\alpha} - 3\beta\frac{\partial G}{\partial\beta}\right)I_1 + 2\frac{\partial G}{\partial\beta}\frac{I_3}{I_2}\right], \tag{3}$$

$$\tau_k^i = \sigma_k^i - \frac{1}{3}\sigma_\alpha^\alpha\delta_k^i = \frac{\rho\Phi}{2I_2G}\left[\left(G - \alpha\frac{\partial G}{\partial\alpha} - 3\beta\frac{\partial G}{\partial\beta}\right)\left(e_k^i - \frac{I_1}{3}\delta_k^i\right) + 2\frac{I_3}{I_2^2}\frac{\partial G}{\partial\beta}\left(e_\alpha^i e_k^\alpha - \frac{1}{3}I_2\delta_k^i\right)\right]. \tag{4}$$

We will consider an incompressible fluid as the limiting case of a compressible fluid. In this case, by virtue of (3), the limit $\lim\limits_{I_1\to 0} I_1\,\partial\ln G/\partial\alpha = \lim\limits_{\alpha\to 0}\sqrt{\alpha}\,I_2(\partial\ln G/\partial\alpha = a$ is finite, and consequently, $\lim\limits_{\alpha\to 0}\alpha(\partial\ln G/\partial\alpha) = 0$; Eqs. (3) and (4) take the form

$$\sigma_\alpha^\alpha = \frac{\rho\Phi}{I_2}\left[3a + \frac{I_3}{I_2}\frac{d\ln g}{d\beta}\right], \tag{3'}$$

$$\tau_k^i = \frac{\rho\Phi}{2I_2}\left[\left(1 - 3\beta\frac{d\ln g}{d\beta}\right)e_k^i + 2\frac{I_3}{I_2^2}\frac{d\ln g}{d\beta}\left(e_\alpha^i e_k^\alpha - \frac{1}{3}I_2\delta_k^i\right)\right], \tag{4'}$$

where $g(\beta) = G(0, \beta)$.

We will compare these expressions with the spur and the deviator of the tensor (1):

$$\sigma_\alpha^\alpha = -3\lambda + 4\eta I_2, \tag{1'}$$

$$\tau_k^i = 2\mu e_k^i + 4\eta\left(e_\alpha^i e_k^\alpha - \frac{1}{3}I_2\delta_k^i\right). \tag{1''}$$

Comparison of (1") and (4') leads to the equations

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$$2\mu = \frac{\rho\Phi}{2I_2}\left(1 - 3\beta\frac{d\ln g}{d\beta}\right), \qquad (5)$$

$$4\eta = \frac{\rho\Phi I_3}{I_2^3}\cdot\frac{d\ln g}{d\beta}, \qquad (6)$$

and from (1'), (3'), and (6) we obtain

$$\lambda = -\rho\Phi a/I_2.$$

For an incompressible fluid, $\lambda = 0$ and, consequently, $a = 0$.

We will now make the limiting transition from an arbitrary flow to a flow with zero third invariant of the tensor e^i_k and we will require that in this case the effect of normal stresses should be preserved. The transverse viscosity then does not vanish, and by virtue of (6) the limit $\lim_{\beta\to 0}\sqrt{\beta}(d\ln g/d\beta) = c$ is finite. Consequently, for small β the asymptotic $g(\beta) \sim g(0)\exp(2c\sqrt{\beta})$ holds, and for $\beta = 0$, Eqs. (5) and (6) take the form

$$2\mu = \rho\Phi/2I_2,\ 4\eta = \rho\Phi c/I_2\sqrt{I_2}.$$

Simple calculations show that in this case for pure shear flow the rate of the difference between the normal stresses to the tangential stress is independent of the shear velocity $(\sigma_2^2 - \sigma_3^3)/\sigma_1^2 = c\sqrt{2}$, which does not agree with experimental data. If both viscosities are constant, relation (1) does not agree with the principle of the maximum rate of entropy production. Indeed, in view of (1)

$$\Phi = \frac{1}{\rho}\sigma^i_k e^k_i = \frac{1}{\rho}(2\mu I_2 + 4\eta I_3),$$

while from the equation ([2], p. 88)

$$\sigma^i_k = \rho\left(\frac{\partial\Phi}{\partial e^i_k}e^k_i\right)^{-1}\Phi\frac{\partial\Phi}{\partial e^i_k}$$

we have

$$\tau^i_k = \frac{\mu I_2 + 2\eta I_3}{\mu I_2 + 3\eta I_3}\left[2\mu e^i_k + 6\eta\left(e^i_\alpha e^\alpha_k - \frac{1}{3}I_2\delta^i_k\right)\right],$$

which contradicts (1").

Hence, Eq. (1) is unsuitable for describing the effect of normal stresses. It can obviously only be used when $\eta = 0$. In this case the viscosity μ is the independent of I_3 ([3], p. 121).

NOTATION

σ^i_k is the nonequilibrium part of the stress tensor;

e^i_k is the deformation rate tensor;

λ, μ, η are the bulk, shear, and transverse viscosities;

δ^i_k is the Kronecker delta.

LITERATURE CITED

1. Rheology (Theory and Applications) [Russian translation], IL, Moscow (1962).
2. H. Ziegler, Extremal Principles of the Thermodynamics of Irreversible Processes and the Mechanics of Continuous Media [Russian translation], Mir, Moscow (1966).
3. B. E. Pobedrya, Lectures on Tensor Analysis [in Russian], Izd. MGU, Moscow (1974).